Recall A parametrized curve $\alpha : I \to S$ is called a parametrized geodesic if $\frac{D}{dt}\alpha' = 0$

- For any vector $w \in T_pS$, there is a unique parametrized geodesic $\alpha_w : (-\varepsilon_1, \varepsilon_2) \to S$ with $\alpha(0) = p$, $\alpha'(0) = w$.

- Smooth dependence on initial conditions says that for any $w \in T_pS$, $\exists\, \delta$ such that we can choose ε_1 and ε_2 uniformly for $|\tilde{w} - w| < \delta$, and moreover the map sending $(\tilde{w}, t) \to \alpha_{\tilde{w}}(t) \in S$ is smooth.

Thm Given $p \in S$, there is an $\varepsilon > 0$ such that for all $w \in T_pS$ with $|w| < \varepsilon$, α_w exists for time 1, and $\underset{\in S}{\alpha_w(1)}$ depends smoothly on w.

Key Lemma: As long as both exist, $\alpha_{rw}(t) = \alpha_w(rt)$, $r > 0$.

$\ulcorner$ let $\beta(t) = \alpha_w(rt)$. Then $\beta'(t) = r\,\alpha_w'(rt)$, so

- $\frac{D\beta'}{dt} = D_{\beta'}\beta' = D_{r\alpha'}r\alpha' = r^2 D_{\alpha'}\alpha' = 0$
- $\beta'(0) = r\,\alpha'(0) = rw$
- $\beta(0) = p$

so by uniqueness of geodesics, $\beta = \alpha_{rw}$, i.e. $\alpha_{rw}(t) = \beta(t) = \alpha_w(rt)$ $\lrcorner$

Proof of theorem:

$\ulcorner$ By SDOIC with $w=0$, $\exists\ \delta, \varepsilon_2$ such that $\alpha_w(t)$ exists and depends smoothly on w for all $|w|<\delta$ and $0<t<\varepsilon_2$. Then for all $|w|<\frac{\delta\varepsilon_2}{2}$

$$\alpha_w(1) = \alpha_{\frac{2w}{\varepsilon_2}}\left(\frac{\varepsilon_2}{2}\right) \qquad \left|\frac{2w}{\varepsilon_2}\right| < \delta$$

which depends smoothly on w. $\lrcorner$

<u>Dfn</u> The exponential map of S at p is the map $\exp_p : \mathcal{U} \longrightarrow S$ sending w to $\alpha_w(1)$, $\mathcal{U} \subseteq T_pS$

where $\mathcal{U}$ is the open subset of $w \in T_pS$ for which α_w exists for time greater than 1.

We'll calculate $d(\exp_p)_0(v)$. This is a linear map $T_0(\mathcal{U}) \longrightarrow T_{\exp_p(0)}S$. To calculate it,

$$T_0(\mathcal{U}) = T_pS \qquad T_{\exp_p(0)}S = T_pS$$

we need a curve $v(t)$ in $\mathcal{U}$; take $v(t) = tv$.

Then

$$d(\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} \alpha_{tv}(1)$$

$$\overset{\text{Key lem}}{=} \frac{d}{dt}\Big|_{t=0} \alpha_v(t)$$

$$= v$$

We conclude

<u>Prop</u> $d(\exp_p)_0$ is the identity, and $\exp_p$ is a local diffeomorphism at 0.

$\ulcorner$ IFT $\lrcorner$

<u>Dfn</u> A normal neighborhood of $p \in S$ is a neighborhood $V \ni p$ such that $\exp_p$ maps some neighborhood of $0 \in T_pS$ diffeomorphically onto V.

<u>Rmk</u>: $\exp : TS \longrightarrow S \times S$

$(p, v) \mapsto \exp_p(v)$

has derivative at 0 in a chart

$$\begin{bmatrix} id & * \\ 0 & id \end{bmatrix}$$

with respect to $T_pTS \simeq T_pS \oplus T_pS$

$\Rightarrow$ local diffeo

$\Rightarrow$ can find

$U \in S$ s.t. $\exists$ ε s.t. exp is a diffeo to $U \times U$.

- Levi-Civita connection

- Length/energy minimizers are geodesics

Geodesics and Distance

Suppose (M,g) is Riem. and ∇ is a metric connection

If $\gamma: I \to M$ minimizes length between its endpoints,

then $\forall$ extensions $\gamma: \underset{s}{(-\varepsilon,\varepsilon)} \times \underset{t}{I} \longrightarrow M$ w/ $\gamma_s|_{\partial I} = \gamma_0|_{\partial I}$)

($\sqrt{\langle \dot\gamma_0, \dot\gamma_0 \rangle} = 1$)

$$\frac{d}{ds} L(\gamma_s) = 0$$

$$0 = \frac{d}{ds}\int_I \sqrt{\langle \dot\gamma, \dot\gamma \rangle}\, dt = \int_I \langle \nabla_s \gamma_t, \gamma_t \rangle$$

$$= \int_I \langle \nabla_t \gamma_s, \gamma_t \rangle + \langle \nabla_s \gamma_t - \nabla_t \gamma_s, \gamma_t \rangle$$

$$= \int_I - \langle \gamma_s, \nabla_t \gamma_t \rangle + (\qquad\qquad)$$

$\Rightarrow$ If $(\ ,\ ) = 0$, then minimizers are ∇-geodesics

- Gauss's lemma

$r : \mathbb{R}^n \longrightarrow \mathbb{R}$

If $q \in M$, $\mathcal{U}$ normal nbhd, $r_p : \mathcal{U} \longrightarrow \mathbb{R}$ is $r_p(q) = r(\exp^{-1}(q))$.

wlog orth.

<u>Lem</u> $|\nabla r_p| = 1$.

<u>Pf</u> Identify $\mathcal{U}$ w/ its image in $T_pM \cong \mathbb{R}^n$, so we're dealing w/ a Riem metric g on $\mathbb{R}^n - 0$.

Let $\partial_r = \frac{x^i}{r} \frac{\partial}{\partial x^i}$ radial v.f. By constr, $\nabla_{\partial_r} \partial_r = 1$.

Note $g_{ij} = \delta_{ij} + o(1)$

(1) $g(\partial_r, \partial_r) = 1$

$\ulcorner$ $\partial_r g(\partial_r, \partial_r) = 2g(\nabla_{\partial_r} \partial_r, \partial_r) = 0$

$\lim_{r \to 0} g(\partial_r, \partial_r) = \frac{x^i x^j}{r^2} (\delta_{ij}) + o(1) = 1$ $\lrcorner$

(2) If ∂_θ tangent vector to S^n, $\partial_\theta = A^i_j x^j \frac{\partial}{\partial x^i}$ for some skew-sym matrix A_{ij}

$g(\partial_r, \partial_\theta) = 0$

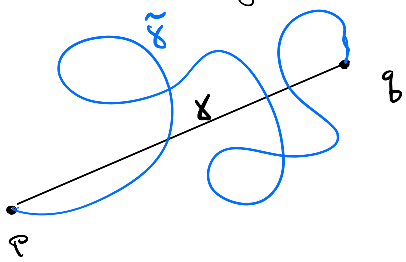
$\ulcorner$ $\partial_r g(\partial_r, \partial_\theta) = \frac{1}{2} \partial_\theta g(\partial_r, \partial_r) = 0$

$\lim_{r \to 0} g(\partial_r, \partial_\theta) = \frac{x^i}{r^2} A^j_k x^k (\delta_{ij} + o(1)) = 0$ $\lrcorner$

- Calibration argument $\Rightarrow$ geodesics are locally minimizing.

Warm-up

Straight lines are minimizing in $\mathbb{R}^2$



Calibration argument:

Find $f: \mathbb{R}^2 \to \mathbb{R}$ such that ★

- $|df| = 1$ (i.e. $df(v) \leq |v| \; \forall v$ and $= |v|$ for some v)
- $df(\dot{\gamma}(t)) = |\dot{\gamma}| \quad \forall t$

Then $L(\gamma) = \int |\dot{\gamma}| = \int df(\dot{\gamma}) = f(q) - f(p)$

and $L(\tilde{\gamma}) = \int |\dot{\tilde{\gamma}}| \geq \int df(\dot{\tilde{\gamma}}) = f(q) - f(p)$ for any $\tilde{\gamma}$

$\Rightarrow \gamma$ is minimizing

Dfn If $|df| = 1$ everywhere, f is a distance function